Depinning from the internal defect in the planar Ising model

1986 J. Phys. A: Math. Gen. 19 L599

(http://iopscience.iop.org/0305-4470/19/10/008)

J. Phys. A: Math. Gen. 19 (1986) L599-L603. Printed in Great Britain

LETTER TO THE EDITOR

Depinning from the internal defect in the planar Ising model

D B Abraham† and N M Švrakić†‡

† Department of Theoretical Chemistry, University of Oxford, 1 Keble Road, Oxford OX1 3NP, UK

‡ Institute of Physics, PO Box 57, 11000 Beograd, Yugoslavia

Received 10 September 1985

Abstract. An exact analysis of depinning of the domain wall from the internal defect in the planar Ising model is given. The model differs from the ones previously studied by having unequal couplings on two sides of the defect. This asymmetry induces a depinning transition at a well defined temperature. The exact phase diagram and incremental free energy are obtained.

The binding of a domain wall by a line of defect bonds in the planar Ising ferromagnet below its Curie temperature T_c has been the subject of much recent study (Abraham 1980, 1981a, b, Burkhardt 1981, Chalker 1981, Kroll 1981, Chui and Weeks 1981, van Leeuwen and Hilhorst 1981, Švrakić 1983). In particular, Abraham (1980, 1981b) has analysed this problem rigorously in considerable detail and has shown that (i) a line of defect bonds in the *interior* of the lattice always binds the wall, while (ii) when the defect is near the *surface* of the lattice, the wall will be bound to it at sufficiently low temperatures and will unbind from the defect when the temperature is increased above a certain value $T_R < T_c$. It is said that an unbinding (depinning, wetting) transition takes place at $T = T_R$. Associated with this transition is a jump in the domain-wall specific heat.

In this letter we propose analysing exactly the binding (and unbinding) of the domain wall by the defect in the *interior* of the planar Ising lattice, but with bonds which differ on two sides of the defect. This problem represents a generalisation of the original unbinding model (Abraham 1980) and cases (i) and (ii) above are obtained in appropriate limits. Thus, by taking various limits we can study within a single framework two seemingly unrelated problems: (i) depinning of the domain wall (Abraham 1980) and (ii) non-universal behaviour of the defect at $T = T_c$ (McCoy and Perk 1980). Moreover, in realistic materials pinning usually occurs at the boundary between two different domains. Our model corresponds to that situation. This has provided motivation for our work.

In order to set up the problem to be solved consider a planar Ising ferromagnet with spins $\sigma(i) = \pm 1$ at all points (i_1, i_2) of a subset Λ of $\mathbb{Z}^2$ of the infinite square lattice with unit side. The lattice is specified so that $\Lambda = \{(i_1, i_2): -N \leq i_1 \leq N-1, -M \leq i_2 \leq M-1\}$. The energy of a spin configuration $\{\sigma\}$ is given by

$$E_\Lambda(\{\sigma\}) = \sum_{|i-j|=1} J(i-j)\sigma(i)\sigma(j) - \sum h(i)\sigma(i) \qquad (1)$$

where $J(k)$ are ferromagnetic (non-negative) couplings and $h(i)$ are external magnetic fields. The probability of the configuration $\{\sigma\}$ is given by

$$p_\Lambda(\{\sigma\}) = Z_\Lambda^{-1} \exp[-\beta E(\{\sigma\})] \qquad (2)$$

with $\beta = 1/k_B T$ (k_B is the Boltzmann constant, T is the absolute temperature) and Z is the partition function which normalises (2). It is convenient to denote $K(i-j) = \beta J(i-j)$.

If $\langle\ \rangle_\Lambda(h, T)$ denotes expectation with respect to (2) then, provided $T < T_c$ (Peierls 1936, Dobrushin 1968, Griffiths 1964, Martin-Löf 1972, Yang 1952, Bennettin *et al* 1973, Abraham and Martin-Löf 1973),

$$\lim_{h\to 0^+} \lim_{\Lambda\to\infty} \langle\sigma(0,0)\rangle_\Lambda(h, T) = m^* \tag{3}$$

where m^* is the spontaneous magnetisation. The same limiting result is obtained when $h(i) = 0$ for all i except those at the boundary $\partial\Lambda$ of the lattice where $h(i) = \infty$ and only configurations with $\sigma(i) = +1$ on $\partial\Lambda$ contribute to (2). $\Lambda\to\infty$ means that $(0, 0)$ becomes infinitely far from the boundary.

In order to study phase separation we will consider boundary conditions B_Λ^{+-} on the lattice Λ so that $\sigma(i) = +1$ (respectively $\sigma(i) = -1$) for $i \in \partial\Lambda$ whenever $i_2 > 0$ (respectively $i_2 < 0$). As $\Lambda\to\infty$ we expect a phase of magnetisation m^* (respectively $-m^*$) far above (below) the line $i_2 = 0$. The domain-wall free energy is defined as

$$\tau = -\lim_{N\to\infty} \lim_{M\to\infty} \frac{1}{2N+1} \ln(Z(B_\Lambda^{+-})/Z(B_\Lambda^{++})] \tag{4}$$

where $Z(B_\Lambda^{++})$ normalises (2) with all boundary spins $\sigma(i) = +1$ ($i \in \partial\Lambda$).

In order to further specialise to the problem at hand the following notation will be used:

$$J|(i_1, i_2); (j_1, j_2)| = \begin{cases} J_1 & \text{for all } i_1, j_1 < 0 \text{ and } i_1 = j_1 = 0 \\ J_2 & \text{for all } i_1, j_1 > 1 \text{ and } i_1 = j_1 = 1 \\ J_0 & \text{for all } i_1 = 0, j_1 = 1 \end{cases}$$

i.e. on one side of the defect (J_0) bonds have strength J_1 and on the other side strength J_2. (This choice is made only to simplify various algebraic equations which follow. We have actually studied a situation in which 'orthogonal' and 'parallel' bonds in the J_1 half-plane are different—likewise for the J_2 half-plane—but this merely increases the complexity of the equations without giving additional insight.) Clearly when $J_1 = J_2$ the model reduces to the case with the internal defect. Such a defect binds the wall at all temperatures $0 < T < T_c$ (Abraham 1981b). Here an interesting phenomenon occurs at $T = T_c$ when the spin-spin correlation function exponent shows continuous dependence on the defect coupling strength.

On the other hand, when, for example, $J_1 = \infty$, the defect is positioned next to the fully ordered system ('surface') (Abraham 1980) from which the domain wall unbinds at a temperature given by equation (6) below (equation (8) of Abraham (1980)). Clearly, if the defect coupling strength $J_0 = J_2$, unbinding occurs at $T = 0$ since, in the absence of the defect, the wall has no preference to be bound for any particular place in the lattice. In what follows we shall assume, without loss of generality, that $J_1 \leq J_2$ and $J_0 = aJ_2$ $(0 \leq a \leq 1)$ (thus $J_0 < J_2 < J_1$). The new result of this letter is that the domain wall unbinds from the *internal* defect whenever $J_1 \neq J_2$ at some temperature $T_R < T_c$, where T_c solves the equation $\sinh^2 2K_2 = 1$. Explicitly, let $K_i = \beta J_i$ $(i = 0, 1, 2)$ and let $b = J_2/J_1$ (by the above assumption $0 \leq b \leq 1$). Then the unbinding transition temperature $T_R(a, b)$ is given as the non-trivial solution of the equation

$$\sinh 2K_1^* + (\sinh^2 2K_1 - 1)/\{\sinh 2K_1 + [c_1^2(\sinh^2 2K_1 - 1)^2 + 1]^{1/2}\} = \cosh 2(K_2 - K_2^*) \tag{5}$$

where

$$c_1 = \cosh 2K_1[(1-A_1)/(1+A_1)]$$

and

$$A_1 = \tanh^2 K_1 \tanh^2 K_2/\tanh^4 K_0.$$

Note that, when $K_1 = \infty$, (5) reduces to (Abraham 1980, equation (8))

$$\tanh^2 K_0 = \tanh K_2 \tanh(K_2 - K_2^*) \qquad (6)$$

where $\exp 2K_i = \coth K_i^*$ $(i = 1, 2)$ is the dual coupling. Also, equation (5) becomes identity when $K_1 = K_2 = 0.5 \sinh^{-1}(1)$ for all K_0.

Equation (5) gives the phase boundary for depinning of the domain wall into the (weaker) J_2 half-plane. This phase boundary is plotted in figure 1 for several values of the parameter b $(b = J_2/J_1)$. Note that only when b becomes quite close to unity does the transition temperature $T_R(a, b)$ begin to approach the critical temperature T_c appreciably. Other limiting behaviours are also apparent in figure 1. (i) When $a = 0$, i.e. when $J_0 = 0$, we have two separate Ising half-planes and boundary conditions are irrelevant. Then $T_R = T_c$ since the spontaneous magnetisation vanishes at T_c. (ii) When $a = 1$, i.e. when $J_0 = J_2$, the defect is of equal strength to the bulk (J_2) and cannot pin the wall at any finite temperature: hence $T_R(1, b)$. (iii) Finally, when $b = 1$, i.e. when $J_1 = J_2$, the symmetric situation on two sides of the defect prevents depinning into any one half-plane (Abraham 1981b) and $T_R(a, 1) = T_c$, which is the temperature at which the difference between two phases disappears.

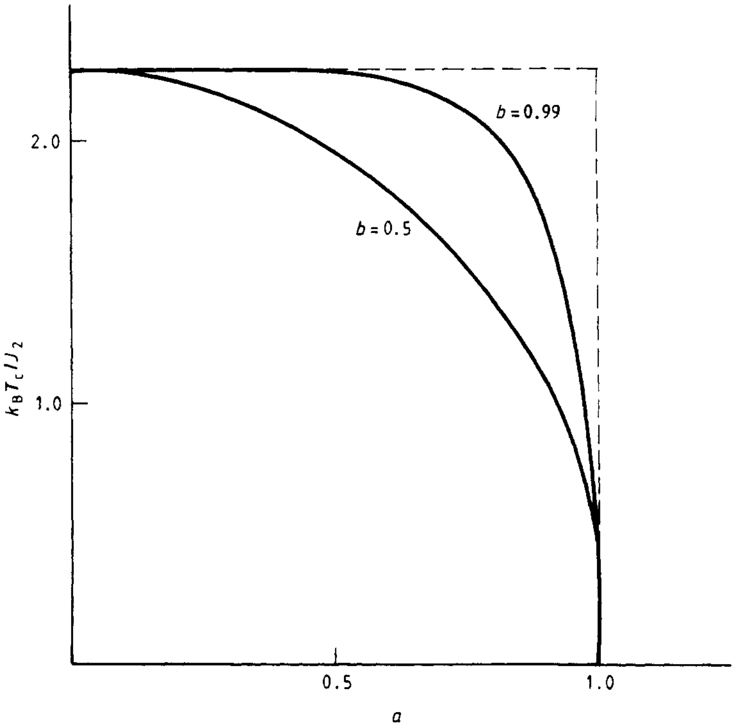

Figure 1. Phase diagram for depinning into the J_2 half-plane. Parameter a $(a = J_0/J_2)$ is along the x axis. Parameter b $(b = J_2/J_1)$ is 0.5 and 0.99. This diagram is to be compared with figure 1 of Abraham (1980).

The incremental free energy τ is calculated from (4). By the use of the transfer matrix theory and recently developed techniques (Abraham 1978a, b, 1980) it can be shown that τ takes the usual Onsager (1944) values for $T_R(a, b) < T < T_c$, but the second temperature derivative, or the domain-wall specific heat, exhibits a jump discontinuity at $T_R(a, b)$. The domain-wall free energy τ has a value $\ln B$ ($B = \tanh K_2^* \coth K_2$) for $T_R(a, b) < T < T_c$ and the value $|v_0|$ for $0 < T < T_R(a, b)$, where v_0 solves the equation

$$\cosh v_0 = 0.5(B + 1/B) + 1 - \cosh \gamma_2(\omega) \tag{7}$$

with

$$\left|\frac{\cosh(2K_1^* - \gamma_1(\omega)) - \cosh 2K_1}{\cosh(2K_1^* + \gamma_1(\omega)) - \cosh 2K_1} \frac{\cosh(2K_2^* - \gamma_2(\omega)) - \cosh 2K_2}{\cosh(2K_2^* + \gamma_2(\omega)) - \cosh 2K_2}\right| = \tanh^2 K_1 \tanh^2 K_2 / \tanh^4 K_0 \tag{8}$$

where $\gamma_i(\omega)$ are Onsager's functions (Onsager 1944) given by

$$\cosh \gamma_i(\omega) = \cosh 2K_i^* \cosh 2K_i - \sinh 2K_i^* \sinh 2K_i \cos \omega \qquad i = 1, 2 \tag{9}$$

and $\exp 2K_i = \coth K_i^*$. Figure 2 shows the domain-wall free energy τ calculated from (7), (8) and (9) for $a = 0.6$, $b = 0.5$ and $b = 0.99$. In the same figure the interfacial free energy ('surface tension') ($a = 1$, $b = 1$) is shown (Onsager 1944) for comparison. Note that τ coincides with 'surface tension' for temperatures $T > T_R(a, b)$. In the limit $b = 1$, i.e. $K_1 = K_2 = K$, the incremental free energy, as given by (7), vanishes at $T_R(1, a) = T_c$ as

$$v_0 = \sigma(K - K_c) \tag{10}$$

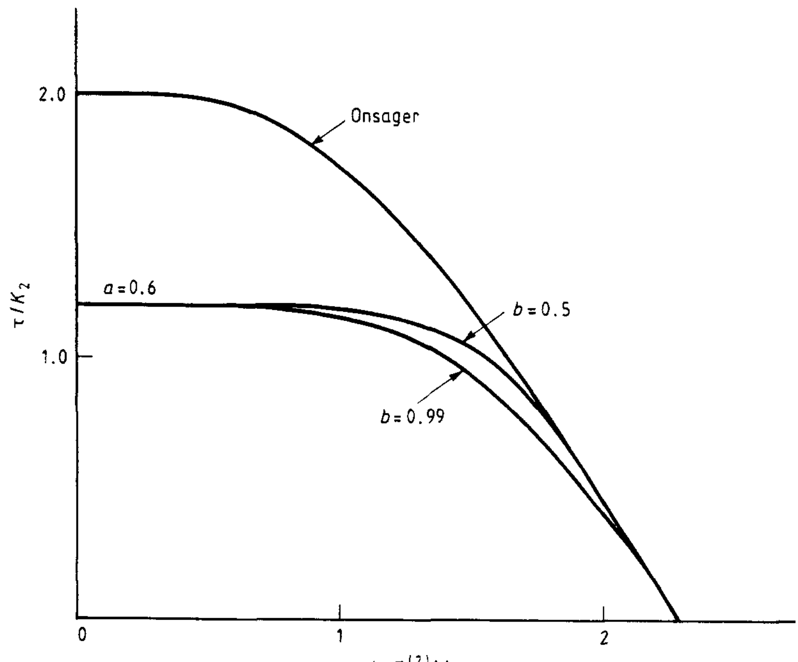

Figure 2. Incremental (defect, domain wall) free energy as a function of temperature. Note that this free energy merges with Onsager's 'surface tension' at $T = T_R$ for $a = 0.6$, from figure 1.

where K_c solves $\sinh 2K_c^* \sinh 2K_c = 1$. We note, firstly, that the domain-wall free energy vanishes linearly as T approaches T_c, which is consistent with scaling ($\mu = 1$) and, secondly, that the coefficient σ depends on the value of the defect coupling (J_0) as

$$\sigma = \sigma_{\text{Onsager}}(1 - \kappa^2)^{1/2} \tag{11}$$

where σ_{Onsager} is the Onsager's amplitude $\sigma_{\text{Onsager}} = 4$ and

$$\kappa = (\tanh^2 K_0 - \tanh^2 K)/(\tanh^2 K_0 + \tanh^2 K) \tag{12}$$

evaluated at $K = K_c$.

The dual of the partition function ratio in (4) is a correlation function for the pair of spins at a distance d along the line of *enhanced* (defect) bonds. At $T = T_c$ (with $b = 1$) it is known (McCoy and Perk 1980, Bariev 1979) that the corresponding exponent η of the correlation function depends continuously on the value of the defect coupling strength. This non-universality is also seen in the amplitude dependence given by (11) and (12). The method presented in this letter enables one to study non-universal behaviour of line defects with considerable ease.

Finally, let us consider a solid-on-solid limit of the problem just solved.

(i) In the case $b = 0$ (i.e. $K_1 = \infty$) the solid-on-solid (SOS) limit corresponds to the quantum mechanical problem of the particle moving in the semi-infinite potential well (i.e. a well of finite depth with one infinite wall).

(ii) In the case $b = 1$ we have a simple well for which there is always a bound state (the interface is always pinned).

(iii) The model we have studied corresponds to the well with unequal (but finite) walls. This SOS version of the problem has been studied elsewhere (Wolff and Švrakić 1984).

References

Abraham D B 1978a *Commun. Math. Phys.* **59** 17
—— 1978b *Commun. Math. Phys.* **60** 181
—— 1980 *Phys. Rev. Lett.* **44** 1165
—— 1981a *Phys. Rev. Lett.* **47** 545
—— 1981b *J. Phys. A: Math. Gen.* **14** L369
Abraham D B and Martin-Löf A 1973 *Commun. Math. Phys.* **32** 245
Bariev R Z 1979 *Sov. Phys.-JETP* **50** 613
Bennettin G, Galavotti G, Jona-Lasinio G and Stella A L 1973 *Commun. Math. Phys.* **30** 45
Burkhardt T W 1981 *J. Phys. A: Math. Gen.* **14** L63
Chalker J T 1981 *J. Phys. A: Math. Gen.* **14** 2431
Chui S T and Weeks J D 1981 *Phys. Rev.* B **23** 2438
Dobrushin R L 1968 *Theor. Prob. Appl.* **13** 197
Griffiths R B 1964 *Phys. Rev.* **136** 437
Kroll D M 1981 *Z. Phys.* B **41** 345
McCoy B M and Perk J H H 1980 *Phys. Rev. Lett.* **44** 840
Martin-Löf A 1972 *Commun. Math. Phys.* **32** 245
Onsager L 1944 *Phys. Rev.* **65** 117
Peierls R E 1936 *Proc. Camb. Phil. Soc.* **32** 477
Švrakić N M 1983 *J. Phys. A: Math. Gen.* **16** L171
van Leeuwen J M J and Hilhorst H J 1981 *Physica* A **107** 319
Wolff W and Švrakić N M 1984 *J. Phys. A: Math. Gen.* **17** 3383
Yang C N 1952 *Phys. Rev.* **85** 808